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*by L. M. Rauch*

Prepared under Grant No. NsG-413 *by*
SETON HALL UNIVERSITY
South Orange, N. J.
*for*

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L. M. RAUCH

ABSTRACT



The regularized N-body problem is resolved in explicit form by the introduction of a pseudo time parameter in the form of a Sundman and Levi-Civita transformation. The solution consists of two phases, namely by the formal representation of the solution by a pseudo time series and the analytical justification of the process. A brief third part is added as a directive in the essential mode of numerization of the problem.

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# INTRODUCTION

This paper resolves the regularized N-body problem in explicit form. It differs from reference [5] in that the system of differential equations are regularized by the introduction of a pseudo time parameter $\sigma$ in the form of a Sundman [6] and Levi-Civita [3] transformation relating $\sigma$ with the dynamic time parameter. This leads to some new results, specifically so relative to the movable singularities in the solution. However, the methodologies intersect in many common regions, so that some essential results of the first paper [5] are applicable to the present one.

The explicit resolution of the regularized problem consists of two phases: (1) the formal representation of the solution by a pseudo time series and (2) the analytical justification of the formal series. The first part of the paper thus generates the general term in the form of an irreducible recursive expression which ultimately is a function of the boundary conditions. The second part carries over, by means of the regularizing transformation, the analytical results generated in ref. [5] to the formal regularized solution. A brief third part is added as a directive to the essential processes involved in the numerical solution of the problem.

## I. THE FORMAL REGULARIZED FORMULAE

This part deals with (1) a summary of some necessary expressions given in reference [5], (2) with the simplification and the regularization of the equations of motion of the n bodies and (3) with the deduction of the formal solution of the regularized motion in terms of a power series of the pseudo time.

### Regularization and Summary of Some Results in [5]:

The classical equations of motion for the n bodies in a potential field v, are given in a Cartesian inertial frame of reference [4] as

$$\ddot{x}^{ih} = \frac{1}{m_i} \frac{\partial v}{\partial x^{ih}}, \quad \ddot{x}^{ih} \equiv \frac{d^2 x^{ih}}{dt^2} \tag{1.1}$$

under the following definitions:

$$i, j = 1,2 - - -, n; \; h = 1,2,3; \; i \neq j$$

$x^{ih}$ : components of the position vector $x^i$ of the i particle (mass $m_i$) relative to an inertial frame.

$X^{ij} = x^j - x^i$ : the relative position vector from the i to the j particle.

$$v = \sum_j \frac{H_i m_j}{\left(R^{ij}\right)^{1/2}}, \quad H_i \equiv G m_i$$

$X^{ijh}$ : the components of the relative vector $X^{ij}$

$R^{ij} \equiv |X^{ij}|^2 = \sum_h \left(x^{ih} - x^{jh}\right)^2 \equiv \sum_h \left(X^{ijh}\right)^2$ : the square of the magnitude of the vector $X^{ij}$

$S^{ij} \equiv \left(R^{ij}\right)^{-\frac{3}{2}}$ : a scalar quantity.

The equations of motion thus take the form,

$$\ddot{x}^{ih} = \sum_j H_j S^{ij} X^{ijh}, \quad i \neq j \tag{1.2}$$

A special regularizing transformation [3,6] involving the parameter $\sigma$, is given by

$$\dot{\sigma} \equiv \frac{d\sigma}{dt} = v \tag{1.3}$$

The application of this transformation to the classical form (1.2) of the equations of motion, leads to the regularized representation,

$$v \frac{d}{d\sigma} \left( v \frac{dx^{ih}}{d\sigma} \right) = \sum_j H_j S^{ij} X^{ijh} \tag{1.4}$$

As is manifest the right member is an invariant under the transformation since the quantities S and X are not explicit functions of the real time t.

Define the following quantities:

$$w^{ih} \equiv v \frac{dx^{ih}}{d\sigma} , \quad W^{ih} \equiv \left(w^{ih}\right)^2 \tag{1.4'}$$

The equations of motion (1.4) are thus given as

$$\frac{dW^{ih}}{d\sigma} = 2\frac{dx^{ih}}{d\sigma} \sum_j H_j S^{ij} X^{ijh} \tag{1.5}$$

Further define the symbols:

$$\frac{dW^{ih}}{d\sigma} \equiv W_1{}^{ih} , \quad \frac{dx^{ih}}{d\sigma} \equiv x_1{}^{ih}, W^{ih} \equiv W_o{}^{ih}, S^{ij} \equiv S_o{}^{ij}, X^{ijh} \equiv X_o{}^{ijh},$$

$$w^{ih} \equiv w_o{}^{ih} \tag{1.6}$$

Since the indices i, j, h remain unchanged by the operations that follow, these entities will be deleted for the time being. The system (1.5) may then be written as

$$W_1 = 2\ x_1\ \bar{H}\ S_o X_o = 2\ x_1 \bar{H}\ y_o \Rightarrow W_1{}^{ih} = 2\ x_1 \sum_j H_j S_o^{ij} X_o{}^{ijh} \tag{1.7}$$

where $S_o X_o \equiv y_o$, $\bar{H} = H_j$

Regularized Power Series Expansion: This section deals with the serial representation of the regularized solution of the N-body dynamical system.

The $r^{th}$ derivative of (1.7) relative to $\sigma$ may, by induction, be shown to lead to the statement

$$W_{r+1} \equiv 2 \frac{d^{r+1}}{d\sigma^{r+1}} \left(x_o \overline{H} y_o\right) = 2 \sum_{s=o}^{r} \binom{r}{s} x_{r-s} \overline{H} y_s , \quad r = 0,1,2,--- \tag{2.1}$$

Expand the $x_{r-s}$ and $y_s$ quantities in a $\sigma$ power series,

$$x \equiv x_o = \sum_{k=o}^{\infty} \xi_k \sigma^k , x_m \equiv \frac{d^m x_o}{d\sigma^m} = \sum_{k=o}^{\infty} m! \binom{k}{m} \xi_k \sigma^{k-m}, m=0,1,2,---,k \geq m \tag{2.2}$$

$$y \equiv y_o = \sum_{k=o}^{\infty} \frac{y_k(0)}{k!} \sigma^k , \quad y_r = \sum_{k=o}^{\infty} \frac{y_k(0)}{(k-T)!} \sigma^{k-r}, \quad r = 0,1,2,---,$$

$$k \geq r, \; y_k(0) = \left(y_k\right)_{\sigma=o} \tag{2.2'}$$

Substitute (2.2) and 2.2') in (2.1),

$$W_{r+1} = 2 \sum_{s=o}^{r} \binom{r}{s} \sum_{k=o}^{\infty} \left(r-s\right)! \binom{k}{r-s} \xi_k \sigma^{k-r+s} \overline{H} \sum_{k=o}^{\infty} \frac{y_k(0)}{(k-s)!} \sigma^{k-s} \tag{2.3}$$

In view of the Cauchy product

$$\sum_{k=o}^{\infty} A_k \sum_{k=o}^{\infty} B_k = \sum_{k=o}^{\infty} \sum_{u=o}^{k} A_{k-u} B_u$$

(2.3) may be written as

$$W_{r+1} = 2 \sum_{s=o}^{r} \sum_{k=o}^{\infty} Q\, \xi_{k-u} \quad \bar{H}\ y_u(0)\sigma^{k-r}, \quad Q = Q(k,r,u,s)$$

$$= \frac{(r-s)!}{(u-s)!} \binom{r}{s} \binom{k-u}{r-s} \tag{2.4}$$

Since the quantities $S \equiv S_o$ and $X \equiv X_o$ are not explicitly dependent on the time parameter t, the formulae for $y_o = S_o X_o$ and its derivatives as given in [5] are valid if the pseudo time parameter $\sigma$ replaces t. Thus the expressions (2.5), (5.2) and (6.20) given in that reference, if applied to (2.4), leads to the statement

$$\frac{y_r^{ijh}}{(r+2)!} = \sum_{p=o}^{r} \sum_{g=1}^{T(p)} (r-p)!\ \frac{G}{H} \left| {ijh \atop rpg} \right| \tag{2.5}$$

The symbols in (2.5) are defined as follows:

(a) $$\left| {ijh \atop rpg} \right| = \eta^{ijh}_{r-p} \left(R^{ij}_o\right)^{-\frac{2p+3}{2}} \prod_{f=o}^{p} \left[ \sum_{n=1}^{3} \sum_{\ell=o}^{\left[\frac{f}{2}\right]} (f,\ell) f! \right.$$

$$\left. \eta^{ijn}_{\ell}\, \eta^{ijn}_{f-\ell} \right]^{\alpha^p_{fg}}$$

(b) $(f,\ell) = 2$ or $1$ according as $\ell < \frac{f}{2}$ or $\ell = \frac{f}{2}$ respectively,

(c) $\left[\frac{f}{2}\right]$ Is the largest integer not exceeding $\frac{f}{2}$

(d) $\alpha^{p}_{fg}$ are the solutions of the linear Diophantine equation

$$\sum_{f=1}^{p} f\, \alpha^{p}_{fg} = p,\ p = 1,2,---,\ g = 1,2,---,\ P(p):$$

$$\text{For } f = 0 \quad \alpha^{p}_{og} = p - \sum_{f=1}^{p} \alpha^{p}_{fg},\ \alpha^{o}_{oo} \equiv 0,\ p = 1,2,---$$

(e) $G = \frac{2}{\Gamma(\gamma)}\ F(a,b;c;1)\ F(\alpha,\beta;\gamma;\ 1)$, ref.[9]

and where the quantities $a,b,c;\ \alpha,\beta,\gamma$ are given by (5.5) and (5.8) in reference [5].

(f) $H \equiv H\left(\alpha^{p}_{fg,f}\right) = \prod_{f=1}^{p} \left(\alpha^{p}_{fg+1}\right) \left[\Gamma(f+1)\right]^{\alpha^{p}_{fg}}$

(g) $\eta^{ijh}_{q} = \xi^{jh}_{q} - \xi^{ih}_{q}$

On applying (2.5) for $r = u$ expression (2.4) is changed to

$$W_{r+1} = 2 \sum_{k=o}^{\infty} \sum_{s=o}^{r} \sum_{u=o}^{k-w} \sum_{v=o}^{w} \sum_{\ell=o}^{r+1} (u+2)!(u-p)!Q\xi_{k-u} \bar{H} \sum_{p=o}^{u} \sum_{g=1}^{T(p)} \frac{G}{H}$$

$$\left|\begin{matrix} ijh \\ upq \end{matrix}\right|_{\sigma=o} \sigma^{k-r} ;Q = Q(k,r,u,s) = \frac{(r-s)!}{(u-s)!}\binom{r}{s}\binom{k-u}{r-s},\ \bar{H} = \sum_{j=1}^{n} Gm_{j} \qquad (2.6)$$

Equation (2.6) expresses $W_{r+1}$, as given by (2.1), in terms of the coefficients ($\xi$'s) of the power series solution (2.2). The next step in the determination of an irreducible recursive formula for the $\xi$'s is to utilize the definition $W \equiv w^2$

By induction we derive from this definition the statement

$$W_{r+1} = \sum_{i=o}^{r+1} \binom{r+1}{i} w_i w_{r+1-i} \qquad (2.7)$$

Define the potential as a power series

$$v \equiv v_o = \sum_{k=o}^{\infty} \gamma_k \sigma^k \qquad (2.8)$$

Change (1.4') to the form

$$w_o = v_o x_1, \; x_1 = \frac{dx}{d\sigma} \qquad (2.9)$$

With the use of (2.2), (2.8) and the Cauchy product formula, (2.9) becomes

$$w_o = \sum_{k=o}^{\infty} \sum_{u=o}^{k} \gamma_{k-u} \xi_u \sigma^k \qquad (2.10)$$

By induction $\dfrac{d^s w_o}{d\sigma^s}$ is shown to be

$$w_s = \sum_{k=o}^{\infty} \sum_{u=o}^{k} s!\binom{k}{s} \gamma_{k-u} \xi_u \sigma^{k-s} \; , \; s = 0,1,2,- - - \qquad (2.11)$$

Substitute (2.11) into (2.7) and again utilize the Cauchy formula. These operations lead to the equation,

$$W_{r+1} = \sum_{\ell=o}^{\infty} \sum_{w=o}^{\ell} \sum_{u=o}^{\ell-w} \sum_{v=o}^{w} (r+1)\binom{\ell}{r+1} \gamma_{\ell-w-u}\gamma_{w-u}\xi_u\xi_v\sigma^{\ell-r-1}, k=\ell \quad (2.12)$$

Expression (2.12) is a second representation of $W_{r+1}$ in terms of the coefficients of (2.8) and the $\xi$'s of (2.2).

## Formulation of the Irreducible Recursive Formula:

The purpose of this section is to generate an irreducible recursive expression for the $\xi$'s.

Equate the right members of (2.6) and (2.12). Let $\ell = k + 1$ and form the equality of the coefficients of like powers of the $\sigma$'s. These operations generate a relation between the $\gamma$ and $\xi$ coefficients, namely,

$$\sum_{w=o}^{k+1} \sum_{u=o}^{k+1-w} \sum_{v=o}^{w} \psi_1(k,r,u,w,v;\gamma,\xi) = \phi\,(u,v,w;\gamma,\xi) \quad (3.1)$$

where

$$\psi_1 = (r+1)!\binom{k+1}{r+1}\gamma_{k+1-w-u}\gamma_{w-u}\xi_u\xi_v \quad (3.2)$$

$$\phi = \sum_{k,s,u,p,g} 2\,(u+2)!(u-p)!\; Q\bar{H}\begin{vmatrix} ijh \\ upg \end{vmatrix} \xi_{k-u}; s=0,1---,r;$$

$$u=0,1,---,k; p=0,1,---,u \quad (3.3)$$

Let v take the specific value w, v = w, so that (3.1) becomes

$$\sum_{w=o}^{k+1} \sum_{u=o}^{k+1-w} (r+1)! \binom{k+1}{r+1} \gamma_{k+1-w-u}\gamma_{w-u}\xi_u\xi_w =$$

$$\phi - \sum_{w=o}^{k+1} \sum_{u=o}^{k+1-w} \sum_{v=o}^{w-1} \psi_1$$

Let w = k + 1 and the above statement becomes

$$\left\{\begin{array}{l} (r+1) \binom{k+1}{r+1} \gamma_o\gamma_{k+1}\xi_o\xi_{k+1} = \phi - \sum_{w=o}^{k+1} \sum_{u=o}^{k+1-w} \sum_{v=o}^{w-1} \psi_1 \\ \qquad - \sum_{w=o}^{k} \sum_{u=o}^{k+1-w} \psi_2 , \\ \psi_2 = (r+1)! \binom{k+1}{r+1} \gamma_{k+1-w-u}\gamma_{w-u}\xi_u\xi_w \end{array}\right. \tag{3.4}$$

Add the left member of (3.4) to both sides of the equation. Thus

$$\left\{\begin{array}{l} 2(r+1)! \binom{k+1}{r+1} \gamma_o\gamma_{k+1}\xi_o\xi_{k+1} = \phi - \sum_{u,v,w} \left(\psi_1 + \psi_2\right) \\ o \leq w \leq k + 1, \; o \leq u \leq k + 1 - w, \; o \leq v \leq w - 1 \end{array}\right. \tag{3.5}$$

It follows that

$$\xi_{k+1} = \frac{\phi - (r+1)!\binom{k+1}{r+1}\sum_{u,v,w}\gamma_{k+r-w-u}\gamma_{w-u}\xi_u\left(\xi_v+\xi_w\right)}{2(r+1)!\binom{k+1}{r+1}\gamma_o\gamma_{k+1}\xi_o} \tag{3.6}$$

The identity (3.6) is a recursive expression of the $\xi$'s in terms of the $\gamma$'s and $\xi$'s with lesser subscripts.

To attain an irreducible recursive form for $\xi$, the $\gamma$'s, as given in (3.6), must be expressed in terms of the $\xi$'s. The serial definition for the potential namely

$$v \equiv v_o = \sum_{p=o}^{\infty} \gamma_p \sigma^p, \gamma_p = \left(\frac{v_p}{p!}\right)_{\sigma=o}, \quad v_p \equiv \frac{d^p v_o}{d\sigma^p}, \tag{3.7}$$

is used for the derivation of the required recursive form. Define

$$T^{ij} \equiv \left(R^{ij}\right)^{-\frac{1}{2}} \tag{3.8}$$

or more generally $T \equiv t_o = R_o^{m}$, $R = R_o$, $m = -\frac{1}{2}$ where the superscripts are again deleted. The expression (3.8) of reference [5] becomes (by replacing $T_p$ for $S_p$ and t by the pseudo time parameter $\sigma$),

$$\left\{\begin{aligned}
&T_p \equiv \frac{d^p T_o}{d\sigma^p} = R_o^{m-p} \sum_{g=1}^{P(p)} D\left(\bar{p},m\right) K\left(p,\alpha^p_{fg}\right) \prod_{f=o}^{p} \left(R_f\right)^{\alpha^p_{fg}} \\
&\sum_{f=1}^{p} f\,\alpha^p_{fg} = p, \alpha^p_{og} = p - \sum_{f=1}^{p} \alpha^p_{fg}, \quad \alpha^o_{oo} = 0, p = 1,2,---, g = 1,2,---,P(p) \\
&D(m,p) = (m-\bar{p})\,\bar{p}!\binom{m}{p}, \quad \bar{p} = p - \alpha^o_o g, \quad p = 1,2,---, \quad m = -\frac{1}{2} \\
&K\left(p,\alpha^p_{fg}\right) = \frac{p!}{\prod_{f=1}^{p}\left(\alpha^p_{fg}\right)!\,(f!)^{\alpha^p_{fg}}}
\end{aligned}\right. \tag{3.9}$$

The value of $R_f$ in terms of the X's is given in ref. [5] by equation (4.2),

$$R_f^{ij} = \sum_{h=1}^{3} \sum_{\ell=o}^{\left[\frac{f}{2}\right]} (f,\ell)\binom{f}{\ell} X_\ell^{ijh} X_{f-\ell}^{ijh} \quad , \; f = 1,2,- - - \tag{3.10}$$

where the parameter $\sigma$ replaces t. In view of the formula

$$v_r = \sum_j H_{ij} T_p^{ij} \quad , \; H_{ij} = Gm_i m_j$$

it follows that

$$v_p = \overline{\overline{H}}\, R_o^{m-p} \sum_{g=1}^{P(p)} DK \prod_{f=o}^{p} \left[\sum_{h,\ell} (f,\ell)\binom{f}{\ell} X_\ell X_{f-\ell}\right]^{\alpha_{fg}^{p}} ,$$

$$\overline{\overline{H}} \equiv H_{ij} \tag{3.11}$$

where the superscripts have been deleted.

By means of the defined expansions

$$X \equiv X_o = \sum_{k=o}^{\infty} \eta_k \sigma^k \quad , \; X_r = \sum_{k=o}^{\infty} r!\binom{k}{r} \eta_k \sigma^{k-r} \; , \; \varepsilon_k^{ijh} \equiv \xi_k^{jh} - \xi_k^{ih} \tag{3.12}$$

the expression for $X_r$, when $\sigma = 0$ is given as

$$\left(X_r\right)_{\sigma=o} = r!\eta_r \tag{3.13}$$

In terms of the superscripts and of $\sigma = 0$, (3.11) becomes

$$\binom{v}{p}_{\sigma=o} = \sum_{j} H_{ij} \left(R_o^{ij}\right)^{m-p} \sum_{g=1}^{P(p)} DK \prod_{f=o}^{p} \left[ \sum_{s=1}^{3} \sum_{\ell=o}^{\left[\frac{f}{2}\right]} (f,\ell) \binom{f}{\ell} X_{\ell}^{ijh} X_{f-\ell}^{ijh} \right]_{\sigma=o}^{\alpha_{fg}^{p}} \quad (3.14)$$

It follows that

$$\gamma_p^{ih} \equiv \left[\frac{v_p}{p!}\right]_{\sigma=o} = \frac{1}{p!} \sum_{j,g} H_{ij} \left(R_o^{ij}\right)_{\sigma=o}^{-\frac{2p+1}{2}} DK \prod_{f=o}^{p} \left[ \sum_{s,\ell} (f,\ell)\, f!\, \eta_{\ell}^{ijs} \eta_{f-\ell}^{ijs} \right]_{\sigma=o}^{\alpha_{fg}^{p}} \quad (3.15)$$

In view of (6.20) of [5] expression (3.15) is written as

$$\gamma_p^{ih} = \frac{\begin{vmatrix} ijh \\ rpg \end{vmatrix}_{\sigma=o}}{\eta_{r-p}^{ijh} \left(R_o^{ij}\right)_{\sigma=o}} \quad (3.16)$$

In combination with (3.6), (3.16) gives the desired irreducible recursive relation for the $\xi$'s.

The formal solution of the regularized system of equations of motion (1.4) or (1.5) expressed in terms of a power series in the pseudo time parameter $\sigma$, is given by equations (2.2), (3.6) and (3.16). Statements (3.6) and (3.16) are the specifications for the coefficients of the series (2.2) as irreducible recursive forms.

To be sure the general coefficient $\xi^{ih}_{k+1}$ is ultimately given in terms of the initial positions $\xi^{ih}_{o}$ and velocities $\dot{\xi}^{ih}_{1}$. To formulate such an explicit statement would lead to unwieldy operations and forms whose utility in numerical evaluation or ease in dynamical interpretation, is highly questionable.

## II. ANALYTICAL PHASE

The analytical aspect of the regularized n body problem follows the formal phase where the coefficients $\xi_k$ of the pseudo time series have been generated as irreducible recursive functions. The basic consideration of the validity of the serial representation of the solution over a time region was shown to hold [5] for all t except for well defined movable singularities. These existential phases for dynamical (real) time will be transformed to considerations of pseudo time.

<u>The Relation Between Pseudo and Real Time:</u> In this section the function $\sigma = \sigma(t)$ is first generated followed by the determination of its inverse.

(a) The regularizing transformation (1.3), $\frac{d\sigma}{dt} = v$, leads to the indefinite integral

$$\sigma = \int u dt + c$$

In view of (3.7) the above integral becomes

$$\sigma = \sum_{p=o}^{\infty} \frac{\left(v_p\right)_{t=o}}{(p+1)!} t^{p+1} + c$$

With the assumption that $\sigma = 0$ when $t = 0$ we get

$$\sigma = \sum_{p=o}^{\infty} \frac{\left(v_p\right)_{t=o}}{(p+1)!} t^{p+1} \tag{4.1}$$

To compute $\left(v_p\right)_{t=o}$ expression (3.16) is available where $t = 0$ replaces $\sigma = 0$, namely

$$\left(v_p\right)_{t=o} = p!\ \gamma_p = p!\ \frac{\left|\begin{matrix} ijh \\ rpg \end{matrix}\right|_{t=o}}{\eta_{r-p}\left(R_o\right)_{t=o}} \tag{4.2}$$

Equation (4.1) in conjunction with (4.2) thus establishes a time integral for $\sigma$, namely $\sigma = \sigma(t)$

(b) To determine the inverse function $t = t(\sigma)$ define

$$V = \frac{1}{v} \tag{4.3}$$

Again use the transformation $\frac{d\sigma}{dt} = v$ in the form $dt = Vd\sigma$.

Consider the inductive formula (3.5) of reference [5], namely

$$S_p = R_o^{m-p} \sum_{g=1}^{P(p)} D(m,\bar{p})\, K\left(p,\alpha_{fg}^{p}\right) \prod_{f=o}^{p} R_f^{\alpha_{fg}^{p}} , \quad p = 1,2,3,---$$

This formula is valid for expression (4.3) if S and R are replaced by V and v respectively and m = - 1. Thus

$$V_p = v_o^{-1-p} \sum_{g=1}^{P(p)} D(-1,\bar{p})\, K\left(p,\alpha_{fg}^{p}\right) \prod_{f=o}^{p} v_f^{\alpha_{fg}^{p}} , \tag{4.4}$$

where the quantities D, K, $\bar{p}$, $\alpha_{fg}^{p}$ are defined by the formulae in the expressions (3.9). For the quantity D,

$$D = D(-1,\bar{p}) = (-1)(-1-1)(-1-2)---(-1-\bar{p}) = (-1)^{\bar{p}+1}(\bar{p}+1)!$$

Write $dt = Vd\sigma$ in the form $t(\sigma) = \int V(\sigma)\, d\sigma + c$

With the specification that $t = 0$ when $\sigma = 0$

$$t(\sigma) = \sum_{k=o}^{\infty} \frac{V_k(0)}{(k+1)!} \sigma^{k+1} \tag{4.5}$$

where the quantity $V_k(0) \equiv (V_k)_{\sigma=o}$ is given by its expression in (4.4). The function $t(\sigma)$ given by (4.5) and (4.4), specifies the real time in terms of the pseudo time.

The Movable Singularities of the Solution: It has been observed in reference [5] that the right members of the equations of motion (1.1) or (1.2) are analytic over the finite complex plane on the condition that the magnitude $\left(R_o^{ij}\right)^{1/2}$ of the relative position vector $X^{ij}$ between the i and j particle is not zero, namely that $i \neq j$. A similar condition holds for the right members of the regularized equations of motion combined with an added situation.

To show this consider the first factor $\frac{dx^{ih}}{d\sigma}$ of the right members of the system of equation.

This may be written as

$$\frac{dx}{d\sigma} = \frac{dx}{dt}\frac{dt}{d\sigma} = \frac{\dot{x}}{v}$$

In view of the regularizing transformation (1.3). The right member of (1.5) thus becomes

$$\frac{2x_1}{v} \sum_j H_j S^{ij} X^{ijh}$$

The only singularities (other than the movable ones) that may occur is due to the condition that the field potential $v = 0$. So that for $i \neq j$ and $v \neq 0$ the above function is analytic over the finite complex plane $\sigma$.

Thus the solution, in view of the existence theorem [1], [2] for a system of differential equations, is thus void of singularities (intrinsic or movable) for some region in the complex $\sigma$ plane.

The movable singularities in the complex t plane is given in reference [5] by the condition (7.6)

$$\left(R_o{}^{ij}\right)_{t=o}^{-\frac{3}{2}} \left[3X_o{}^{ijh} R_1{}^{ij} - 2R_o{}^{ij} X_1{}^{ijh}\right]_{t=o} = 0$$

or the two conditions

$$\left(R_o{}^{ij}\right)_o^{-1} = 0 \quad \text{or} \quad \left(3X_o{}^{ijh} \bar{R}_1{}^{ij} - 2R_o{}^{ij} \bar{X}_1{}^{ijh}\right)_o = 0, \tag{5.1}$$

Where the barred letters are used, for the moment, to indicate derivatives relative to the time t.

The regularizing transformation $\dot{\sigma} \equiv \frac{d\sigma}{dt} = v$ is used on the above two conditions to transform them in terms of $\sigma$. Thus

$$\bar{X}_1 = \frac{dX_o}{dt} = \frac{dX_o}{d\sigma}\frac{d\sigma}{dt} = \frac{dX_o}{d\sigma} v = X_1 v \; ; \; X_1 = \frac{dX_o}{d\sigma}$$

$$\bar{R}_1 = \frac{dR_o}{dt} = \frac{dR_o}{d\sigma}\frac{d\sigma}{dt} = \frac{dR_o}{d\sigma} v = R_1 v \; ; \; R_1 = \frac{dR_o}{d\sigma}$$

The two statements for movable singularities relative to t thus turn into the conditions

$$\left(R_o{}^{ij}\right)_{\sigma=o}^{-1} = 0 \quad \text{or} \quad (v)_{\sigma=o} = 0 \quad \text{or} \quad \left(3X_o{}^{ijh} R_1{}^{ij} - 2R_o^{ij} \; X_1{}^{ijh}\right)_{\sigma=o} = 0 \tag{5.2}$$

The first and third movable singular conditions(for $t = 0$) in (5.2) has been discussed in reference [5]. For the added equation $(v)_{\sigma=o} = 0$, a brief discussion is given.

We may either deal with it on the basis of the definition of the potential v or preferably by the use of the regularized equations of motion in the form (1.4). Since the left member is zero for $(v)_{\sigma=0} = 0$, the expression

$$\left[\sum_{j=1}^{n} H_j S^{ij} X^{ijh}\right]_{\sigma=0} \equiv \left[\sum_{j} H_j \left(R^{ij}\right)^{-\frac{3}{2}} X^{ijh}\right]_{\sigma=0} = 0 \tag{5.3}$$

Another form of (5.3) may be generated by the definition of the quantity $\theta_h^{ij}$ as the direction angle of the relative position vector $X^{ij}$ between the i and j masses. With the definition of $R^{ij}$ in mind, namely as the square of the magnitude of the relative position vector $X^{ij}$, the formula

$$\cos\theta_h^{ij} = \frac{X^{ijh}}{\left(R^{ij}\right)^{1/2}} \tag{5.4}$$

is generated. Expression (5.3) then takes the form

$$\sum_{j=1}^{n} \frac{H_j \cos\theta_h^{ij}}{R_o^{ij}} = 0 \text{ for } \sigma = 0;\ i \neq j;\ i,j = 1,2,- - -,n;$$

$$h = 1,2,3 \tag{5.5}$$

Two possibilities unfold: (1) $\cos\theta_h^{ij} = 0$ for any $j \neq i$ and (2) $R_o^{ij} = \infty$ for any j and $\cos\theta_h^{ij} \neq 0$ for any h and j.

For case (1) an added restriction must be imposed, namely the identity

$$\cos^2 \theta^{ij}_{h_1} + \cos^2 \theta^{ij}_{h_2} + \cos^2 \theta^{ij}_{h_3} = 1, \; h_1 \neq h_2, \neq h_3, \; h_1, h_2, h_3 = 1, 2, 3$$

The initial angles must be chosen for a particular j for which the above identity is satisfied. The question whether actual dynamical situations exist under these singular restrictions will, for the time being, be held in obeyance.

For case (2) only the initial relationship between the positions of the n bodies are given. Thus the relationship is independent of the initial velocities. It is in a primitive sense manifest that if the bodies are initially mutually infinite they will remain so for any finite time. However, this "physical intuition" should be analytically verifiable, namely, that no finite discernable dynamics is possible. We make the verification with the purpose of illustrating some of the analysis.

Let one of the masses, say $m_i$, be in a finite region and the remaining ones at infinity. Since the second case of the movable singular condition implies that $\left(\frac{1}{R_o^{ij}}\right)_{\sigma=o} = 0$

for any $j \neq i$, it follows from (3.16) that $\gamma^{ih}_p = 0$, $p = 1,2,$---

Since $\frac{1}{R_o^{ij}}$ is contained as a factor in the symbol $\left|{ijh \atop rpg}\right|$ given

by (2.5), the $\left|{ijh \atop rpg}\right| = 0$ when $\frac{1}{\left(R_o^{ij}\right)_{\sigma=o}} = 0$. So that **from** (3.6)

$$\xi_{k+1}^{ih} = 0 \quad \text{when} \quad \frac{1}{\left(R_o^{\ ij}\right)_{\sigma=o}} = 0,\ k = 0,1,2,- - -$$

The expansion for x in (2.2) becomes

$$x = \xi_o + \xi_1 \sigma \tag{5.6}$$

Use (4.1) to transform (5.6) to real time,

$$x^{ih} = \xi_o^{\ ih} + \xi_1^{\ ih} \sum_{p=o}^{\infty} \frac{\left(v_p\right)_{t=o}}{(p+1)!} t^{p+1}$$

and by (4.2) $\left(v_p\right)_{t=o} = 0$, since the factor $\frac{1}{R_o^{ij}} = 0$ when

$t = \sigma = 0$

So that

$$x^{ih} = \xi_o^{\ ih}, \tag{5.7}$$

$h = 1,2,3$; i is some chosen value over the interval

$1,2,---,n$

The conclusion is manifest; the chosen particle $m_i$ remains in the same position relative to an inertial frame for all finite time regardless of the states of the remaining n-1 bodies at infinity. The existence of movable singularities, given by the second condition of (5.2), implies a degeneration of the n body problem to a single body one with no dynamical states and indeterminate ones of the remaining n-1 bodies in the absolute.

## III. COMPUTATIONAL PROCEDURE AND SUMMARY

The computational procedures for the regularized solution of the n body problem are in the broad aspects the same as those discussed in reference [5]. We list the formulae to be used for the computation and then very briefly specify the numerical process.

Formula for Computation: We list the basic formulae and the definitions of some of the symbols involved. The remaining definitions may be found in the text or in reference [5].

$$x^{ih} \equiv x_o^{ih} = \sum_{k=o}^{\infty} \xi_k^{ih}\sigma^k \ , \ i \neq j, \ i = 1,2,- - -,n; \qquad (2.2)$$

$$h = 1,2,3$$

$$\xi_{k+1}^{ih} = \frac{\phi - (r+1)!\binom{k+1}{r+1} \sum_{u,v,w} \gamma_{k+r-w-u}\gamma_{w-u}\xi_u\left(\xi_u+\xi_w\right)}{2\ (r+1)\binom{k+1}{r+1}\gamma_o\gamma_{k+1}\xi_o} \ ; \qquad (3.6)$$

$$0 \leq u \leq k+1-w, 0 \leq v \leq w-1, 0 \leq w \leq k+1,$$

$$k = 0,1,2,---, \ r = 0,1,2,---$$

$$\gamma_p^{ih} = \frac{\left|{ijh \atop rpg}\right|_{\sigma=o}}{\eta_{r-p}^{ijh}\left(R_o^{ij}\right)_{\sigma=o}} \ , \ p = 1,2,--- \text{ and special} \qquad (3.16)$$

value p = 0, g = 1,2,---, P(p)

$$(a)\quad \left|\begin{matrix} ijh \\ rpg \end{matrix}\right|_{\sigma=o} = \eta^{ijh}_{r-p} \left(R^{ij}_{o}\right)^{-\frac{2p+3}{2}} \prod_{f=o}^{p} \left[ \sum_{n=1}^{3} \sum_{\ell=o}^{\left[\frac{f}{2}\right]} (f,\ell) f! \; \eta^{ijn}_{\ell}\, \eta^{ijn}_{f-\ell} \right]^{\alpha^{p}_{fg}} \tag{2.5}$$

$$(d)\quad \begin{cases} \sum\limits_{f=1}^{p} f\alpha^{p}_{fg} = p, \quad p = 1,2,---, \quad g = 1,2,---P(p) \\ \\ \alpha^{p}_{og} = p - \sum\limits_{f=1}^{p} \alpha^{p}_{fg}, \quad \alpha^{o}_{o\,o} \equiv 0, \quad p = 1,2,--- \end{cases} \tag{2.5}$$

The remaining symbols and some others are given in the text by (2.5), (b), (e), (f), (g).

The above expressions are the basic ones in computation. To these must be added some supplementary formulae and definitions.

$$\phi \equiv \sum_{k,s,u,p,g} 2(u+2)!(u-p)!\; Q\overline{H}\; \frac{G}{H} \left|\begin{matrix} ijh \\ upg \end{matrix}\right|_{\sigma=o} \xi^{ih}_{k=u} ;$$

$$s = 0,1,---,r; \; u = 0,1,---k; \; p = 0,1,---,u; \tag{3.3}$$

$$Q = Q\,(k,r,u,s) = \frac{(r-s)!}{(u-s)!}\binom{r}{s}\binom{k-u}{r-s}, \quad \overline{H} = \sum_{j=1}^{n} m_j G$$

(e) $G = \dfrac{2}{\Gamma(\gamma)} F(a,b;c;1)\ F(\alpha,\beta;\gamma;1)$ where F is a (2.5)
hypergeomtric function and a,---; ,--- are given as specified in text.

$$\text{(f)}\quad H = \prod_{f=1}^{p} \Gamma\left(\alpha^{p}_{fg}+1\right) \left[\Gamma(f+1)\right]^{\alpha^{p}_{fg}} \tag{2.5}$$

To express the pseudo time $\sigma$ in terms of the real time t:

$$\sigma = \sum_{p=o}^{\infty} \frac{\left(v_p\right)_{t=o}}{(p+1)!}\ t^{p+1} \tag{4.1}$$

$$\left(v_p\right)_{t=o} = p!\ \gamma_p^{ih} \frac{\left|{}^{ijh}_{rpg}\right|_{t=o}}{\eta^{ijh}_{r-p} \left(R_o^{ij}\right)_{t=o}} \tag{4.2}$$

The movable singularities of the solution satisfy the following three conditions:

$$\left(R_o^{ij}\right)^{-1}_{\sigma=o} = 0,\ \left(v\right)_{\sigma=o} = 0,\ \left(3X_o^{ijh}R_1^{ij}-2R_o^{ij}X_1^{ijh}\right)_{\sigma=o} = 0 \tag{5.2}$$

The second condition of (5.2) may also be written as

$$\cos\theta_h^{ij} = \frac{X_o^{ijh}}{\left(R_o^{ij}\right)^{1/2}} \tag{5.4}$$

The Initial Computational Procedure: The initial computations on which the remaining numerical phases depend is the evaluation of the quantities symbolized by $\begin{vmatrix} ijh \\ rpg \end{vmatrix}$ and given by (2.5), (a). This in turn depends on the "matrices of solutions" $\alpha^{p}_{fg}$ of the Diophantine linear equation expressed by (2.5), (d). Our object in this section is to specify very briefly these two numerical modes. A somewhat more elaborate specification of the unregularized numerication is given in reference [5].

1. The quantities $\alpha^{p}_{fg}$ and $\alpha^{p}_{og}$ satisfy the Diophantine and linear equations (2.5), (d). Tabular matrices for the $\alpha$'s are constructed. As an illustration consider the values $p = 1, 2---,5$ and $p = 0$.

Tabular Matrix for p = 1, 2---, 5 and p = 0

| f \ g | p = 0 | p = 1 | p = 2 | p = 3 | p = 4 | p = 5 |
|---|---|---|---|---|---|---|
| $\alpha^{p}_{og}$ | 0 | 0 | 0 1 | 0 1 2 | 0 1 2 2 3 | 0 1 2 2 3 3 4 |
| $\alpha^{p}_{1g}$ | | 1 | 2 0 | 3 1 0 | 4 2 1 0 0 | 5 3 2 1 1 0 0 |
| $\alpha^{p}_{2g}$ | | | 0 1 | 0 1 0 | 0 1 0 2 0 | 0 1 0 2 0 1 0 |
| $\alpha^{p}_{3g}$ | | | | 0 0 1 | 0 0 1 0 0 | 0 0 1 0 0 1 0 |
| $\alpha^{p}_{4g}$ | | | | | 0 0 0 0 1 | 0 0 0 0 1 0 0 |
| $\alpha^{p}_{5g}$ | | | | | | 0 0 0 0 0 0 1 |

2. To evaluate the quantity $\begin{vmatrix} ijh \\ rpg \end{vmatrix}$ a simplification is made by the following formulation. Define

$$\phi(f,\ell,i,j,s) \equiv \sum_{\ell=0}^{\left[\frac{f}{2}\right]} f!(f,\ell) \sum_{s=1}^{3} n_\ell n_{f-s} \quad \text{where } i,\ j,\ s$$

is deleted for the time being;

$$\psi(p,---) = \prod_{f=0}^{p} \left[\phi(f,\ell,---)\right]^{\alpha^p_{fg}}$$

It follows, on the basis of these definitions, that

$$\psi(p,---) \doteq \left[\phi(p,\ell,---)\right]^{\alpha^p_{pg}} \cdot \prod_{f=0}^{p-1} \left[\phi(f,---)\right]^{\alpha^p_{fg}}$$

Evaluate the $\phi$'s initially and from these the $\psi$'s may be determined.

$$\phi(0,---) = 0!\ (0,0) \sum_s n_o^2 = \sum_s n_o^2$$

$$\phi(1,---) = 1!\ (1,0) \sum_s n_o n_1 = \sum_s 2 n_o n_1$$

$$\phi(2,---) = 2! \left[(2,0) \sum_s n_o n_1 + (2,1) \sum_s n_1^2\right] = 2!$$

$$\left[2 \sum_s n_o n_2 + \sum_s n_1^2\right]$$

$$\phi(3,---) = 3! \left[(3,0) \sum_s n_o n_3 + (3,1) \sum_s n_1 n_2\right] = 3!$$

$$\left[2 \sum_s n_o n_3 + 2 \sum_s n_1 n_2\right]$$

$$\phi(4,---) = 4! \left[2 \sum n_o n_4 + 2 \sum n_1 n_s + \sum n_2^{\,2}\right]$$

$$\phi(5,0---) = 5! \left[2 \sum n_o n_5 + 2 \sum n_1 n_4 + 2 \sum n_2 n_3\right]$$

The expressions

$\begin{vmatrix} ijh \\ rpg \end{vmatrix} = Q^{ijh}_{rp} \quad \psi(p,---)$ become

$\begin{vmatrix} ijh \\ rog \end{vmatrix} = Q^{ijh}_{ro} \quad \psi(0,---) = Q^{ijh}_{ro}.1$ (since by definition $\alpha^o_{oo} = 0$)

$$\begin{vmatrix} ijh \\ r1g \end{vmatrix} = Q^{ijh}_{r1} \quad \psi(1,---) = Q^{ijh}_{r1} \left[\phi(0,---)\right]^{\alpha^1_{og}}$$

$$\left[\phi(1,---)\right]^{\alpha^1_{1g}}$$

$$\begin{vmatrix} ijh \\ r2g \end{vmatrix} = Q^{ijh}_{r2} \quad U(2,---) = Q^{ijh}_{r2} \left[\phi(0,---)\right]^{\alpha^2_{og}}$$

$$\left[\phi(1,---)\right]^{\alpha^2_{1g}} \left[\phi(2,---)\right]^{\alpha^2_{2g}}$$

$$\left|{ijh \atop r3g}\right| = Q^{ijh}_{r3}\ \psi(3,---) = Q^{ijh}_{r3} \left[\phi(0,---)\right]^{\alpha^3_{og}} \left[\phi(1,---)\right]^{\alpha^3_{1g}}$$

$$\left[\phi(2,---)\right]^{\alpha^3_{2g}} \left[\phi(3,---)\right]^{\alpha^3_{3g}}$$

The two basic phases in numerization of the regularized n body problem is fulfilled with an indication of the processes involved in formulae (2.5), (a) and (2.5), (d). The remaining processes necessary in the determination of the coefficients $\xi^{ih}_{k}$, $k = 2,3,---$ of the $\sigma$ series (2.2) in terms of the initial conditions given by $\xi^{ih}_{o}$ and $\xi^{ih}_{1}$ are readily attainable. However, some of the necessary definitions, not listed at the beginning of this section, may be found in the text proper or in reference [5].

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